# QUADRATIC DIFFERENTIALS $A(z-a)(z-b)\,dz^2/(z-c)^2$ AND ALGEBRAIC CAUCHY TRANSFORM

Mohamed Jalel Atia, Gabes.Tunisia,
Faouzi Thabet, Gabes.Tunisia

*Abstract.* In this paper, we discuss the representability almost everywhere (a.e.) in $\mathbb{C}$ of an irreducible algebraic function as the Cauchy transform of a signed measure supported on a finite number of compact semi-analytic curves and a finite number of isolated points. We discuss the existence of critical trajectories of a family of quadratic differentials $\dfrac{A(z-a)(z-b)}{(z-c)^2}dz^2$.

*Keywords*: Algebraic equation. Cauchy transform. Quadratic differentials.

## 1. Introduction

We remind the problem set by B. Shapiro [10]: Is it true that if there exists a signed measure whose Cauchy transform satisfies an irreducible algebraic equation a.e. in $\mathbb{C}$ then there exists, in general, another signed measure whose Cauchy transform satisfies a.e. in $\mathbb{C}$ the same algebraic equation and whose support is a finite union of compact curves and isolated points? Does there exist such a measure with singularity on each connected component of its support?

The aim of this paper is to solve the above problem in the case of algebraic equation

$$p(z)\,h^2(z)-q(z)\,h(z)+r=0, \tag{1.1}$$

where $p$ and $q$ are polynomials of degree 1, and $r\in\mathbb{C}^*$.

More precisely, we will investigate the existence of a compactly supported positive measure whose Cauchy transform coincides with (a branch of) an analytic continuation of a solution $h(z)$ of equation (1.1) a.e. in $\mathbb{C}$. If such a real measure exists and its support is a finite union of compact semi-analytic curves and isolated points we will call it a *real motherbody measure* of (1.1). Recall that the *Cauchy transform* $\mathcal{C}_\mu$ of a compactly supported finite complex-valued Borel measure $\mu$ is the analytic function defined by

$$\mathcal{C}_\mu(z)=\int_{\mathbb{C}}\frac{d\mu(t)}{z-t},\quad z\in\mathbb{C}\backslash\operatorname{supp}(\mu).$$

For instance, if $P$ is a polynomial of degree $n$, then the Cauchy transform $\mathcal{C}_P$ of its normalized root-counting measure $\frac{1}{n}\sum_{p(a)=0}\delta_a$ where $\delta_a$ is the Dirac measure supported at $a$, is given by the formula

$$\mathcal{C}_p(z)=\frac{P'(z)}{nP(z)}=\sum_{p(a)=0}\frac{1}{z-a}.$$

The Cauchy transform $\mathcal{C}_\mu(z)$ satisfies the properties

$$\mathcal{C}_\mu(z)\sim\frac{\mu(\mathbb{C})}{z},\quad z\longrightarrow\infty,\quad \mu=\frac{1}{\pi}\frac{\partial\mathcal{C}_\mu}{\partial\bar{z}}.$$

A special case of equation (1.1) is

$$zh^2(z)+(-z+A)h(z)+1=0, \tag{1.2}$$

which appears in the study of the normalized root-counting measure $\mu_n$,

$$\mu_n=\mu(p_n)=\frac{\sum_{p_n(z)=0}\delta_z}{n}$$

of the rescaled generalized Laguerre polynomials with varying parameters $nA$:

$$p_n(z)=L_n^{\alpha_n}(nz)=\sum_{k=0}^{n}\begin{pmatrix}n+nA\\ n-k\end{pmatrix}\frac{(-z)^k}{k!},$$

with $A<-1$ in [1], and $A\notin\mathbb{R}$ in [9]. It is shown in [2] that the Cauchy transform of the weak limit $\mu$ of $\mu_n$ satisfies equation (1.2), and the support of the measure $\mu$ consists of the trajectories of a certain quadratic differential connecting the zeros $a,b=A+2\pm2\sqrt{A+1}$ of the discriminant of equation (1.2).

Solutions of equation (1.1) are given by

$$h(z)=\frac{q(z)-\sqrt{D(z)}}{2p(z)},$$

with some branch cut of the square root of the discriminant

$$D(z)=q^2(z)-4rp(z)=A(z-a)(z-b),A\in\mathbb{C}^*,(a,b)\in\mathbb{C}^2.$$

It is obvious that with the choice of the square root of $D$ with condition

$$\sqrt{D(z)}\sim q(z),z\to\infty,$$

there exists $\alpha\in\mathbb{C}$ such that $h(z)\sim\frac{\alpha}{z},z\to\infty$.

We begin our study by giving the following necessary conditions for the existence of the real motherbody measure.

**Proposition 1.1.** *If equation (1.1) admits a real motherbody measure $\mu$, then:*

- *any connected curve in the support of $\mu$ coincides with a horizontal trajectory of the quadratic differential*
$$\varpi = -\frac{D(z)}{p^2(z)} dz^2.$$
- *the support of $\mu$ should include both branching points of (1.1) i.e. the zeros of $D$.*

*Proof.* See e.g.[10] or [5]. □

Proposition 1, connects the motherbody measure with horizontal trajectories of a quadratic differential. Quadratic differentials appear in many areas of mathematics and mathematical physics such as orthogonal polynomials, moduli spaces of algebraic curves, univalent functions, asymptotic theory of linear ordinary differential equations etc...

Let us discuss some properties of horizontal trajectories of the rational quadratic differential $\varpi = -\frac{D(z)}{p^2(z)} dz^2$ on the Riemann sphere $\hat{\mathbb{C}}$.

Zeros and simple poles of $-\frac{D(z)}{p^2(z)} dz^2$ are called *finite critical points*, poles of order greater than two are called *infinite critical points.* All other points are called *regular points.*

The *horizontal trajectories* (or just trajectories) of the quadratic differential $\varpi$ are given by the equation

$$\Re \int^z \frac{\sqrt{D(t)}}{p(t)} dt \equiv const. \tag{1.3}$$

The *vertical* or *orthogonal* trajectories are obtained by replacing $\Re$ by $\Im$ in the equation above.

The local structure of the trajectories is well known (see e.g.[8] ,[7], [4], [3]). At any regular point, the trajectory passing through this point is a close analytic arc. Through every regular point of $\varpi$ passes uniquely determined horizontal and vertical trajectories, which are orthogonal to each other [8, Theorem 5.5]. At a zero of multiplicity $r$, there emanate $r+2$ trajectories under equal angles $\pi/(r+2)$. At a simple pole there emanates only one trajectory. At a double pole, the local behaviour of the trajectories depends on the vanishing of the real or imaginary part of the residue; they have either the radial, the circular or the log-spiral form, see Figures 1,2.

A trajectory of $\varpi$ starting and ending at finite critical points is called *finite critical* or *short.* If it starts at a finite critical point but tends either to the origin or to infinity, we call it *an infinite critical trajectory* of $\varpi$.

The set of finite and infinite critical trajectories of $\varpi$ together with their limit points (critical points of $\varpi$) is called the *critical graph* of $\varpi$.

By a translation of the variable $z$ and the change of variable $\sqrt{A}z = y$, we may assume, without loss of generality that,

$$\varpi = \varpi(z, a, b) = -\frac{(z-a)(z-b)}{z^2} dz^2, \ (a, b) \in \mathbb{C}^2 - \{(0,0)\}.$$

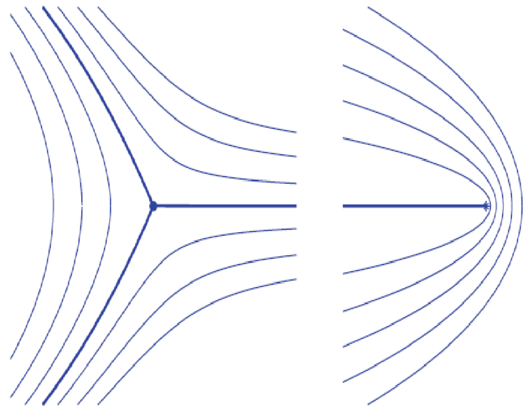

FIGURE 1. The local trajectory structure near a simple zero (left) or a simple pole (right)

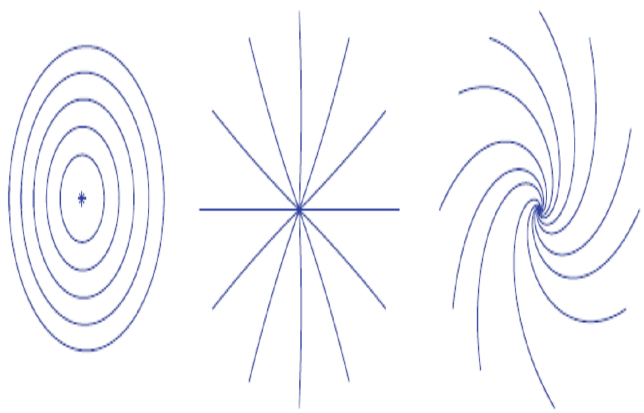

FIGURE 2. The local behaviour of the trajectories near the origin, $ab > 0$ (left), $ab < 0$ (centre), and $ab \notin \mathbb{R}$, (right).

We start by observing that $\varpi$ has two zeros, $a$ and $b$, and, if $ab \neq 0$, the origin is a double pole, with

$$\varpi = \left(-\frac{ab}{z^2} + \mathcal{O}(z^{-1})\right) dz^2, \quad z \to 0.$$

Another pole of $\varpi$ is located at infinity and is of order 4. In fact, with the parametrization $u = 1/z$, we get

$$\varpi = \left(-\frac{1}{u^4} + \mathcal{O}(u^{-3})\right) du^2, \quad u \to 0.$$

If $a = 0$ or $b = 0$, the origin is a simple pole.

Regarding the behavior at infinity, we can assume that the imaginary (resp. real) axis is the only asymptotic direction of the trajectories (resp. orthogonal trajectories) of $\varpi$. In other words, there exists a neighborhood of infinity $U$ such that every trajectory entering $U$ tends to $\infty$ either in the $+i\infty$ or $-i\infty$ direction, and the two rays of any trajectory which stays in $U$ tend to $\infty$ in the opposite asymptotic directions ([8, Theorem 7.4]).

Usually, the main troubles in the description of the global structure of the trajectories of a quadratic differential come from the existence of the so-called recurrent trajectories, whose closures may have a non-zero plane Lebesgue measure. However, since $\varpi$ has only two poles ( 0 and $\infty$ ), Jenkins' Three Pole Theorem asserts that it cannot have any recurrent trajectory (see [8, Theorem 15.2]).

- If $a = b$ then $\varpi = -\frac{(z-a)^2}{z^2} dz^2$, and then there are 4 trajectories emanating from $a$ under equal angles $\pi/2$,
  - if $a \in \mathbb{R}$, two of them diverge to infinity parallel to the imaginary axis in opposite directions; the two others, form a loop around the origin. In case $a = 1$, we get the well-known Szegő curve, see Figure 3.

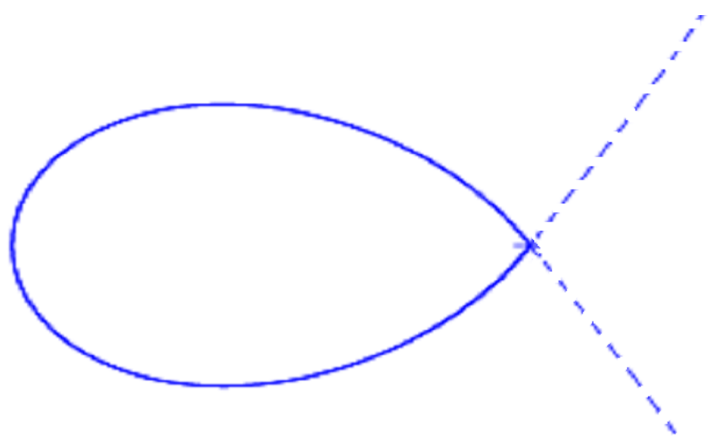

FIGURE 3. Critical graph for the case $a = b = 1$; and the Szegő curve (solid line)

  - if $a \in i\mathbb{R}$, then, then the critical graph is composed by one of the sets $\{iy; y \in \mathbb{R}^+\}$ or $\{iy; y \in \mathbb{R}^-\}$ that contains $a$, and the two other trajectories diverge to infinity and form with infinity a domain that contains 0.
  - if $a \notin \mathbb{R} \cup i\mathbb{R}$, then one spiral trajectory diverges to the origin in , two trajectories diverge to infinity in the same direction and form with infinity a domain that contains the spiral, the fourth trajectory diverges to infinity in the other direction.
- If $a = 0$ and $b \neq 0$, then $\varpi = -\frac{z-b}{z} dz^2$, and there emanate 3 trajectories from $b$ under equal angles $2\pi/3$, one of them goes to the origin, the two others go to infinity parallel to the imaginary axis and in the opposite directions, see Figure 4.

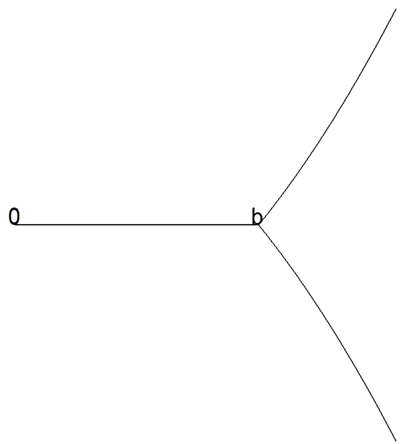


FIGURE 4. Critical graph for the case $a = 0; b > 0$.

In what follows, we investigate $a \neq b$, and $ab \neq 0$. In this case, from each zero, $a$ and $b$, there emanate 3 trajectories under equal angles $2\pi/3$. The local behavior of the trajectories near the origin depends on the vanishing of the real or the imaginary part of the product $ab$.

The main result of this paper is the following.

**Proposition 1.2.** *Let $a$ and $b$ be two non vanishing complex numbers. There exists a critical trajectory of the quadratic differential $\varpi(z,a,b)$ if and only if $\left(\sqrt{a}+\sqrt{b}\right)^2 \in \mathbb{R}$ or $\left(\sqrt{a}-\sqrt{b}\right)^2 \in \mathbb{R}$.*

**Remark 1.3.** (i) *Results of Proposition 2 hold in the case of the rescaled generalized Laguerre polynomials with varying parameters $nA$. In this case we have $a, b = A + 2 \pm 2\sqrt{A+1}$ and*

$$\left(\sqrt{a}\pm\sqrt{b}\right)^2 = 2A + 4 \pm 2\sqrt{((A+2)^2 - 4(A+1)} = 2A + 4 \pm 2\sqrt{A^2}.$$

*In other words $\left(\sqrt{a}\pm\sqrt{b}\right)^2$ equals either $4A+4$ or $4$.*

(ii) *Given a complex number $a$, we consider the set*

$$\Gamma_a = \left\{ b \in \mathbb{C} \mid \left(\sqrt{a}+\sqrt{b}\right)^2 \in \mathbb{R} \text{ or } \left(\sqrt{a}-\sqrt{b}\right)^2 \in \mathbb{R} \right\}.$$

*Straightforward calculations show that if $a \notin \mathbb{R}$, then $\Gamma_a = \mathcal{P}_1 \cup \mathcal{P}_2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are the parabolas defined by :*

$$\begin{aligned}
\mathcal{P}_1 &= \left\{ (x,y) \in \mathbb{R}^2 \mid \Re(a) + 2\left(\frac{y - \Im(a)}{2\Im\left(\sqrt{a}\right)}\right)\Re\left(\sqrt{a}\right) + \left(\frac{y - \Im(a)}{2\Im\left(\sqrt{a}\right)}\right)^2 = x \right\}, \\
\mathcal{P}_2 &= \left\{ (x,y) \in \mathbb{R}^2 \mid \Re(a) - 2\left(\frac{y - \Im(a)}{2\Re\left(\sqrt{a}\right)}\right)\Re\left(\sqrt{a}\right) - \left(\frac{y - \Im(a)}{2\Im\left(\sqrt{a}\right)}\right)^2 = x \right\}.
\end{aligned}$$

*If $a > 0$, then $\Gamma_a = \mathbb{R}^+ \cup \left\{ (x,y) \in \mathbb{R}^2 \mid x = a - \frac{y^2}{4a} \right\}$.*
*If $a < 0$, then $\Gamma_a = \mathbb{R}^- \cup \left\{ (x,y) \in \mathbb{R}^2 \mid x = a - \frac{y^2}{4a} \right\}$.*

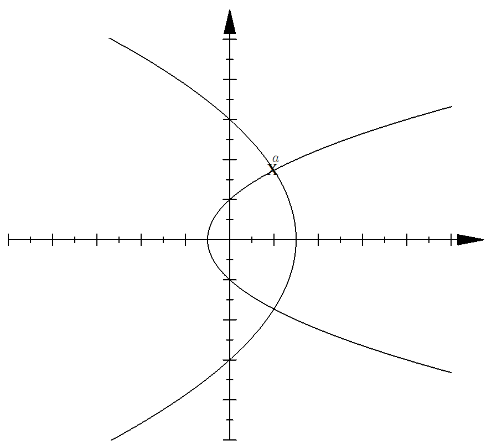

FIGURE 5. Set $\Gamma_a$ when $a \notin \mathbb{R}$.

**Acknowledgements 1.4.** *The authors are grateful to Professor Marco Bertola for useful discussions, and they acknowledge the contribution of the anonymous referee whose careful reading of the manuscript helped to improve the presentation.*

## 2. Proof of Proposition 2.

To prove Proposition 2. we need some lemmas. Below, given an oriented Jordan curve $\Gamma$ joining $a$ and $b$ in $\mathbb{C}^*$, for $t \in \Gamma$, we denote by $\sqrt{D(t)}_+$ and $\sqrt{D(t)}_-$ the limits from the $+$-side and $-$-side respectively. (As usual, the $+$-side of an oriented curve lies to the left, and the $-$-side lies to the right, if one traverses the curve according to its orientation.)

**Lemma 2.1.** *For any curve $\gamma$ joining $a$ and $b$ and not passing through $0$, we have :*

$$\int_\gamma \frac{\left(\sqrt{D(z)}\right)_+}{z} dz = \pm \frac{i\pi}{2}\left(\sqrt{a} \pm \sqrt{b}\right)^2,$$

*the signs $\pm$ depend on the homotopy class of $\gamma$ in $\mathbb{C}^*$, and the branch of the square root $\sqrt{D(z)}$ defined in $\mathbb{C}\backslash\gamma$ is chosen so that $\sqrt{D(z)} \sim z, z \to \infty$.*

*Proof.* With the above choices, consider $I = \int_\gamma \frac{\left(\sqrt{D(z)}\right)_\pm}{z} dz$. Since $\left(\frac{\sqrt{D(t)}}{t}\right)_+ = -\left(\frac{\sqrt{D(t)}}{t}\right)_-$ for $t \in \gamma$, we have

$$2I = \int_\gamma \left[\left(\frac{\sqrt{D(t)}}{t}\right)_+ - \left(\frac{\sqrt{D(t)}}{t}\right)_-\right] dt = \oint_\Gamma \frac{\sqrt{D(z)}}{z} dz,$$

where $\Gamma$ is a closed contour encircling the curve $\gamma$ once in the clockwise direction and not encircling $z = 0$. After a contour deformation we pick up residues at $z = 0$ and at $z = \infty$ for the calculation of $I$, namely :

$$2I = \pm 2i\pi \left(\underset{z=0}{Res}\left(\frac{\sqrt{D(z)}}{z}\right) + \underset{z=\infty}{Res}\left(\frac{\sqrt{D(z)}}{z}\right)\right).$$

Clearly,

$$\underset{z=0}{Res}\left(\frac{\sqrt{D(z)}}{z}\right) = \sqrt{D(0)} = \sqrt{ab}.$$

The residue at $\infty$ is the opposite of the coefficient of $\frac{1}{z}$ in the Laurent serie of $\frac{\sqrt{D(z)}}{z}$. Since $\frac{\sqrt{D(z)}}{z} \sim 1, z \to \infty$, we have

$$\frac{\sqrt{D(z)}}{z} = 1 - \frac{a+b}{2}\frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right);$$

and therefore

$$\underset{z=\infty}{Res}\left(\frac{\sqrt{D(z)}}{z}\right) = \frac{a+b}{2}.$$

$\square$

As an immediate consequence of Lemma 2.1 we get

**Corollary 2.2.** *If $\left(\sqrt{a}+\sqrt{b}\right)^2 \in \mathbb{R}$ or $\left(\sqrt{a}-\sqrt{b}\right)^2 \in \mathbb{R}$, then, there cannot exist two horizontal trajectories emanating from $a$ and $b$ and diverging simultaneously to the origin.*

*Alternatively, if $\left(\sqrt{a}+\sqrt{b}\right)^2 \in i\mathbb{R}$ or $\left(\sqrt{a}-\sqrt{b}\right)^2 \in i\mathbb{R}$, then, there cannot exist two vertical trajectories emanating from $a$ and $b$ and diverging simultaneously to the origin.*

*Proof.* Assume that $ab \notin \mathbb{R}$, and let $\gamma_a$ and $\gamma_b$ be two trajectories that diverge in spirals to the origin. Let $\sigma$ be an orthogonal trajectory that diverges to the origin. Then $\sigma$ intersects $\gamma_a$ and $\gamma_b$ infinitely many times. Considering three consecutive points of intersection, it is

obvious that we can construct two paths $\gamma, \gamma'$ joining $a$ and $b$ and not homotopic in $\mathbb{C}^*$, formed by the three pieces, from $\gamma_a, \sigma$ and $\gamma_b$. Then we get

$$\Re \int_{\gamma} \frac{\left(\sqrt{D(z)}\right)_+}{z} dz \neq 0, \text{ and } \Re \int_{\gamma'} \frac{\left(\sqrt{D(z)}\right)_+}{z} dz \neq 0,$$

which contradicts Lemma 2.1.

If $ab < 0$, the loop formed by vertical trajectories passes through only one zero, either $a$ or $b$; we can repeat the same proof as in the previous case. □

**Definition 2.3.** *A domain in $\mathbb{C}$ bounded only by segments of horizontal and/or vertical trajectories of $\varpi$ (and their endpoints) is called $\varpi$-polygon.*

We can use the Teichmüller lemma (see [4, Theorem 14.1]) to clarify some facts about the global strucure of the trajectories.

**Lemma 2.4** (Teichmuller). *Let $\Omega$ be a $\varpi$-polygon, and let $z_j$ be the singular points of $\varpi$ on the boundary $\partial\Omega$ of $\Omega$, with multiplicities $n_j$, and let $\theta_j$ be the corresponding interior angles with vertices at $z_j$, respectively. Let*

$$\beta_j = 1 - \theta_j \frac{n_j + 2}{2\pi}.$$

*Then*

$$\sum \beta_j = \begin{cases} 0, \textit{if } 0 \in \Omega, \\ \\ 2, \textit{if } 0 \notin \Omega. \end{cases}$$

Any $\varpi$-polygon made of horizontal trajectories and containing the origin can be bounded either by two critical trajectories starting and ending at $a, b$, or it must contain $\infty$ at its boundary and at least one inner angle $\dfrac{4\pi}{3}$.

**Corollary 2.5.** *In the latter case there are a priori three possibilities:*

- *either $\Omega$ is bounded by two critical arcs emanating from the same zero of $\varpi$ and forming an angle $\dfrac{4\pi}{3}$, encircling the origin and going to $\infty$ in the same direction, or*
- *$\Omega$ is bounded by two critical arcs emanating from the same zero of $\varpi$ and forming an angle $\dfrac{2\pi}{3}$, encircling the origin and the other zero, and going to $\infty$ in the same direction, or*
- *$\Omega$ is bounded by two critical arcs emanating from different zeros of $\varpi$ and forming an angle $\dfrac{4\pi}{3}$ going to $\infty$ in the opposite directions.*

**Corollary 2.6.** *There cannot exist two horizontal, or vertical trajectories emanating from the same zero $a$ or $b$ and diverging (radially or spirally) to the origin.*

*Proof.* If there emanate two trajectories from $a$ or $b$ diverging to the origin, consider an $\varpi$-polygon formed by their pieces and a piece of an orthogonal trajectory that diverges to the origin. Clearly this $\varpi$-polygon violates Lemma 2.4. $\square$

**Corollary 2.7.** *Assume that there is no critical trajectory of $\varpi$, then:*
*if ($ab \notin \mathbb{R}$, or $ab < 0$), we get,*

- *either, there exists one trajectory diverging to the origin, four trajectories diverge to infinity in the same direction, and one trajectory diverges to infinity in the other direction.*
- *or, from each zero $a$ and $b$, there emanates one trajectory diverging to the origin, and two trajectories diverge to infinity in the opposite directions.*

*If $ab > 0$, then, from one zero, there is a loop encircling the origin, the third trajectory diverges to infinity. All trajectories emanating from the other zero diverge to infinity, two of them, in the same direction and form with infinity a domain that contains the loop, the third one diverges to infinity in the opposite direction, see Figure 6.*

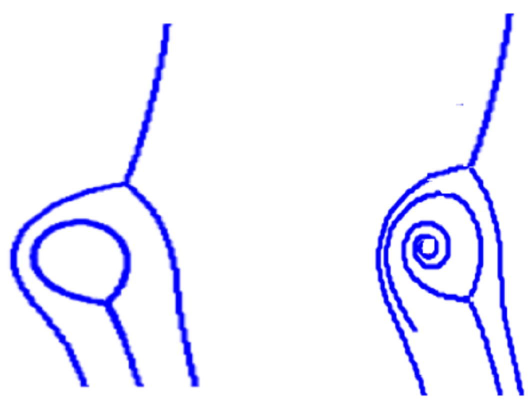

FIGURE 6. Critical graph when $(\sqrt{a} \pm \sqrt{b})^2 \notin \mathbb{R}$, $ab > 0$ (left), $ab \notin \mathbb{R}$ (right).

*Proof.* If $\left(\sqrt{a} \pm \sqrt{b}\right)^2 \notin \mathbb{R}$ and $ab \notin \mathbb{R}$, then, by Lemma 2.1 there is no critical trajectory, and by Corollary 2.6, from each zero, there emanates at most one trajectory that diverges to the origin. Suppose that all trajectories emanating from a zero (for instance $a$ ) diverge to infinity, then, by Lemma 2.4, two of them, say $\gamma_1, \gamma_2$ diverge in the same direction and form, with infinity, a domain $\mathcal{D}$ which contains the origin. By Lemma 2.4, the third trajectory emanating from $a$, cannot diverge to infinity in the same direction as $\gamma_1, \gamma_2$. Corollary 2.5 implies that the interior angle of $\mathcal{D}$ between $\gamma_1$ and $\gamma_2$ equals $\frac{2\pi}{3}$. Then, the domain $\mathcal{D}$ must contain the origin and the other zero $b$.
All these considerations show that there emanate two trajectories from $b$ with the angle $\frac{4\pi}{3}$ that diverge to infinity in the direction of $\gamma_1, \gamma_2$, and which form, with infinity, a domain containing the origin. The third trajectory emanating from $b$ diverges to the origin.
The remaining cases are settled in a similar way $\square$

*Proof of Propostion 2.* (1) If $\left(\sqrt{a}\pm\sqrt{b}\right)^2\in\mathbb{R}$, then, $ab>0$ and we have a loop around 0. By a change of variable $z=\overline{y}$, we see that the critical graph of $\varpi$ is symmetric with respect to the real axis. Thus it follows that

- either, $a,b\in\mathbb{R}$ such that $ab>0$ in which case, the segment $[a,b]$ belongs to the critical graph, and the loop passes through exactly one zero. We have totally two critical trajectories.
- or $a=\overline{b}$ and the loop passes through $a$ and $b$, and, again, we have two critical trajectories [1], see Figure 7.

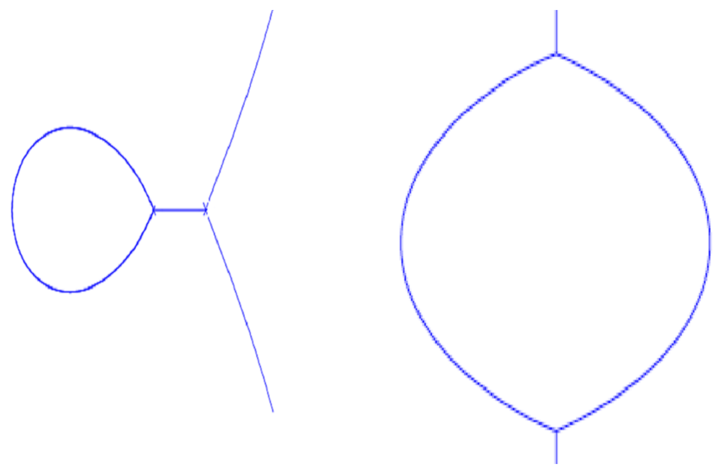

FIGURE 7. Critical graph when $(\sqrt{a}\pm\sqrt{b})^2\in\mathbb{R}$, $a,b>0$ (left), $b=\bar{a}$ (right).

(2) If $\left(\sqrt{a}+\sqrt{b}\right)^2\in\mathbb{R}$ and $\left(\sqrt{a}-\sqrt{b}\right)^2\notin\mathbb{R}$ then $ab\notin\mathbb{R}$ or $ab<0$.
Suppose that $ab\notin\mathbb{R}$. By Corollaries 2.2, 2.6 and 2.7, the critical graph of $\varpi$ possesses exactly one trajectory that emanates from a zero, say $a$ and diverges to the origin. From the other zero $b$, there emanate two trajectories, say $\gamma_1,\gamma_2$, diverging to infinity in the same direction and they form, with infinity, a domain that contains the origin and $a$. From $a$, there emanates at least one vertical trajectory $\sigma_1$ that diverges to infinity (parallel to the real axis). This trajectory must intersects $\gamma_1$ or $\gamma_2$ at some point $M$. Let $\gamma$ be a path connecting $a$ and $b$ in $\mathbb{C}^*$, formed by a piece from $a$ to $M$ of $\sigma_1$ and another one from $M$ to $b$ of $\gamma_1$ or $\gamma_2$. It follows that

$$\begin{aligned}\Re\left(\int_\gamma\frac{\sqrt{D(z)}}{z}dz\right)&=\Re\left(\int_a^M\frac{\sqrt{D(z)}}{z}dz\right)\neq0,\\ \Im\left(\int_\gamma\frac{\sqrt{D(z)}}{z}dz\right)&=\Im\left(\int_M^b\frac{\sqrt{D(z)}}{z}dz\right)\neq0.\end{aligned}\tag{2.1}$$

By Corollary 2.6, there emanates at least one vertical trajectory $\sigma_2$ from $a$, which, either, connects $a$ to $b$, or, it diverges to infinity intersecting $\gamma_1$ or $\gamma_2$ at some point $N$.

The first case contradicts equations (2.1) and the fact that $\left(\sqrt{a}+\sqrt{b}\right)^2 \in \mathbb{R}$. Let $\gamma'$ be a path connecting $a$ and $b$ in $\mathbb{C}^*$ formed by a piece from $a$ to $N$ of $\sigma_2$ and another one from $N$ to $b$ of $\gamma_1$ or $\gamma_2$. Clearly, by lemma 2.4, $\gamma$ and $\gamma'$ are not homotopic in $\mathbb{C}^*$ and equations (2.1) are valid with $\gamma'$. This contradicts the fact that $\left(\sqrt{a}+\sqrt{b}\right)^2 \in \mathbb{R}$. We conclude that a critical trajectory exists. The case $ab < 0$ can be settled in the same way. See Figure 8.

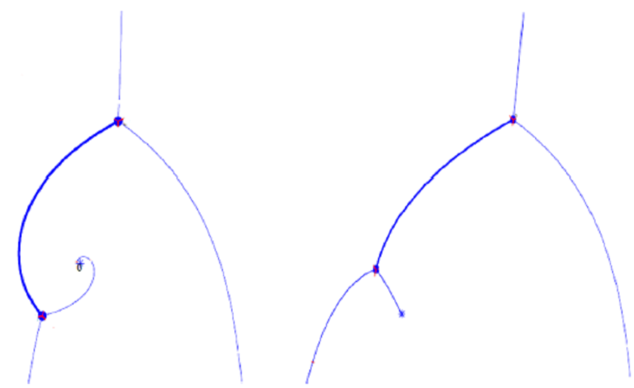

FIGURE 8. Critical graph when $(\sqrt{a}+\sqrt{b})^2 \in \mathbb{R}$ and $(\sqrt{a}-\sqrt{b})^2 \notin \mathbb{R}$ , $ab \notin \mathbb{R}$ (left), $ab < 0$ (right).

(3) If $\left(\sqrt{a}\pm\sqrt{b}\right)^2 \notin \mathbb{R}$, then by Lemma 2.1,

$$\Re\left(\int_a^b \frac{\sqrt{D(z)}}{z}dz\right) \neq 0,$$

for any path of integration in $\mathbb{C}^*$ and there is no critical trajectory.

□

## References


[1] A. B. J. Kuijlaars, K. T.-R. McLaughlin, Asymptotic zero behavior of Laguerre polynomials with negative parameter, Constructive Approximation 20 (4) (2004) 497-523.

[2] A. Martinez-Finkelshtein, P. Martinez-Gonzalez, R. Orive, On asymptotic zero distribution of Laguerre and generalized Bessel polynomials with varying parameters, J. Comp. Appl. Math. 133 (2001), 477–487.

[3] A. Vasil'ev, Moduli of families of curves for conformal and quasiconformal mappings, Vol. 1788 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2002.

[4] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.

[5] Igor E. Pritsker. How to find a measure from its potential. http://arxiv.org/abs/1307.5457.

[6] I. E. Pritsker, R. S. Varga, The Szegő curve, zero distribution and weighted approximation, Trans. Amer. Math. Soc. 349 (10) (1997) 4085–4105. doi:10.1090/S0002-9947-97-01889-8. URL http://dx.doi.org/10.1090/S0002-9947-97-01889-8.

[7] J. A. Jenkins, Univalent functions and conformal mapping, Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 18. Reihe: Moderne Funktionentheorie, Springer-Verlag, Berlin, 1958.

[8] K. Strebel, Quadratic differentials, Vol. 5 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], Springer-Verlag, Berlin, 1984.

[9] M. J. Atia, A. Martinez-Finkelshtein, P. Martinez-Gonzalez, and F. Thabet, Quadratic differentials and asymptotics of Laguerre polynomials with varying complex parameters, J. Math. Anal. Appl. 416 (2014), 52–80.

[10] Rikard Bøgvad and Boris Shapiro. On Motherbody Measures with Algebraic Cauchy Transform. http://arxiv.org/abs/1406.1972

*Authors' addresses*:

*Mohamed Jalel Atia*, Faculté des sciences de Gabès, Cité Erriadh Zrig 6072 Gabès Tunisia e-mail: jalel.atia@gmail.com.

*Faouzi Thabet*, ISSAT de Gabès, Avenue Omar Ibn El Khattab 6029 Gabès Tunisia e-mail: faouzithabet@yahoo.fr.